Homework 6 Solutions

provided by Caleb

Extra Exercise 1

(a) Let X be a Bernoulli RV with distribution fcn.

$$p(0)=1-p \quad \text{and} \quad p(1)=p \quad p\in[0,1]$$

$$E(x)=\sum x P(X=x) = 0(1-p)+1(p)=p$$

$$\boxed{\Longrightarrow E(X)=p}$$

$$\text{var}(X)=E[X^2]-(E[X])^2$$
$$=0^2(1-p)+1^2\cdot p - p^2$$
$$=p-p^2=p(1-p)$$

$$\boxed{\text{var}(X)=p(1-p)}$$

(b) Let X be a binomial RV w/ parameters (n,p)

$$P(S)=p$$

Let $X=\sum_{i=1}^{n} X_i$ where $X_1,\ldots,X_n$ are independent Bernoulli Trials.

$$\Longrightarrow E[X]=E\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n} E[X_i]=np \text{ from (a) above.}$$

Since each X_i is independent,

$$\text{var}(X)=\text{var}\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n}\text{var}(X_i)=np(1-p)$$

12.4.2 #24

$$S=\{1,2,3,\ldots,n\} \quad P(X=k)=\frac{1}{n},\ k\in S$$

(a) $$E[X]=\sum_{x=1}^{n} x p(x)=\sum_{x=1}^{n} x\left(\frac{1}{n}\right)=\frac{1}{n}\sum_{x=1}^{n} x=\frac{1}{n}\left(\frac{n(n+1)}{2}\right)=\boxed{\frac{n+1}{2}}$$

(b) $$\text{Var}(X)=E[X^2]-(E[X])^2$$
$$=\sum_{x=1}^{n} x^2 p(x)-\left(\frac{n+1}{2}\right)^2=\frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{(n+1)^2}{4}$$
$$=\frac{2(n+1)(2n+1)}{12}-\frac{3(n+1)^2}{12}=\frac{n+1(4n+2-3n-3)}{12}\boxed{=\frac{(n+1)(n-1)}{12}}$$

12.4.3 #34

$P(\text{heads}) = 0.3 \qquad P(\text{Tails}) = 0.7$

Let X be number of tails $\quad n=5$

(a) $P(X=2) = \binom{5}{2}(0.7)^2(0.3)^3$

(b) $P(X\geq 1) = 1 - P(X=0) = 1-(0.3)^5$

12.5.1 #2

$$f(x) = \begin{cases} \frac{1}{2} & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

$f(x)$ is a density function $\iff \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} 0\,dx + \int_0^2 \frac{1}{2}\,dx + \int_2^{\infty} 0\,dx$$

$$= 0 + \left(\frac{x}{2}\right)\Big|_0^2 + 0 = 1 \quad \therefore f(x) \text{ is a density fcn.} \square$$

Notice for $x \leq 0$, $\int_{-\infty}^{x} f(u)\,du = 0$

for $0<x<2$, $\int_0^x f(u)\,du = \frac{x}{2}$

for $x \geq 2$ $\int_{-\infty}^{x} f(u)\,du = 1$

$$\Rightarrow F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x}{2} & \text{for } 0<x<2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

12.5.1 #4

This question will not be graded. #6 will be graded, see solution below.

$$f(x) = \begin{cases} \frac{c}{x^2} & \text{for } x>1 \\ 0 & \text{for } x \leq 1 \end{cases} \qquad \text{Find } c \text{ s.t. } \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} 0\,dx + \lim_{a\to\infty} \int_1^a \frac{c}{x^2}\,dx = \lim_{a\to\infty} -\frac{c}{x}\Big|_1^a = 0 - \left(-\frac{c}{1}\right) = c = 1 \iff c = 1$$

$\therefore c = 1$

SOLUTION TO 12.5/EXERCISE 6

PROVIDED BY MONA

The distribution function of X is given as $f(x) = \frac{1}{2}e^{-|x|}$. As any function defined in terms of absolute values, this is a function defined piecewise:

$$f(x) = \begin{cases} \frac{1}{2}e^{x} & \text{if } x < 0, \\ \frac{1}{2}e^{-x} & \text{if } x \geq 0. \end{cases}$$

By definition, $E(X) = \int_{-\infty}^{\infty} xf(x)dx$. Note that we would have to split up this integral somewhere anyway in order to take the limits at $-\infty$ and ∞ separately. Since the function is defined differently on the intervals $(-\infty, 0)$ and $(0, \infty)$, we have to integrate different functions in those ranges, we might as well split up our integral at 0. Thus,

$$E(X) = \int_{-\infty}^{0} \frac{1}{2}xe^{x}dx + \int_{0}^{\infty} \frac{1}{2}xe^{-x}dx.$$

Let's compute each piece.

$$\begin{aligned} \int_{-\infty}^{0} xe^{x}dx &= \lim_{a\to-\infty} \int_{a}^{0} xe^{x}dx \\ &= \lim_{a\to-\infty} (xe^{x} - e^{x})\Big|_{a}^{0} \\ &\quad \text{(used integration by parts with substitution } u = x \ du = dx; v = e^{x} \ dv = e^{x}dx.) \\ &= \lim_{a\to-\infty} (-1 + e^{a} - ae^{a}) \\ &= -1 + \lim_{a\to-\infty} e^{a} - \lim_{a\to-\infty} ae^{a}. \end{aligned}$$

Now

$$\lim_{a\to-\infty} e^{a} = 0$$

but the second limit is of indeterminate form, so we rewrite it as a limit of the form $\frac{\infty}{\infty}$ and use L'Hopital to compute it

$$\lim_{a\to-\infty} ae^{a} = \lim_{a\to-\infty} \frac{a}{e^{-a}} \overset{\frac{\infty}{\infty}}{=} \lim_{a\to-\infty} \frac{1}{-e^{-a}} = 0.$$

(Note that in this last line, when we are taking the limit as $a \to -\infty$, we are regarding ae^{a} as a function of a, so a is the variable we are differentiating with respect to when we apply L'Hopital.)

Thus

$$\int_{-\infty}^{0} xe^{x}dx = -1 \quad \text{and} \quad \int_{-\infty}^{0} \frac{1}{2}xe^{x}dx = -\frac{1}{2}.$$

Now let's compute the second piece.

$$\begin{aligned}
\int_0^\infty xe^{-x}dx &= \lim_{b\to\infty}\int_0^b xe^{-x}dx \\
&= \lim_{b\to\infty}(-xe^{x}-e^{-x})\Big|_0^b \\
&\quad \text{(used integration by parts with substitution } u=x\ du=dx; v=-e^{-x}\ dv=e^{-x}dx.) \\
&= \lim_{b\to\infty}(-be^{-b}-e^{-b}+1) \\
&= 1-\lim_{b\to\infty}e^{-b}-\lim_{b\to\infty}be^{-b}.
\end{aligned}$$

Again,

$$\lim_{b\to\infty}e^{-b}=0,$$

and using L'Hopital we find that

$$\lim_{b\to\infty}be^{-b}=\lim_{b\to\infty}\frac{b}{e^b}\overset{\frac{\infty}{\infty}}{=}\lim_{b\to\infty}\frac{1}{e^b}=0.$$

Thus,

$$\int_0^\infty xe^{-x}dx=1 \quad \text{and} \quad \int_0^\infty \frac{1}{2}xe^{-x}dx=\frac{1}{2}$$

and

$$E(X)=-\frac{1}{2}+\frac{1}{2}=0.$$

The variance is given by

$$\text{var}(X)=E(X^2)-E(X)^2,$$

therefore we need to compute $E(X^2)$. Recall that for any real valued function g,

$$E(g(X))=\int_{-\infty}^\infty g(x)f(x)dx,$$

therefore in this case, for $g(x)=x^2$, we get

$$E(X^2)=\int_{-\infty}^0\frac{1}{2}x^2e^xdx+\int_0^\infty\frac{1}{2}x^2e^{-x}dx.$$

Again we can compute each piece.

$$\begin{aligned}
\int_{-\infty}^0 x^2e^xdx &= \lim_{a\to-\infty}\int_a^0 x^2e^xdx \\
&= \lim_{a\to-\infty}(x^2e^x-2xe^x+2e^x)\Big|_a^0 \\
&\quad \text{(used integration by parts twice)} \\
&= \lim_{a\to-\infty}(1-a^2e^a+2ae^a-2e^a) \\
&= 2-\lim_{a\to-\infty}a^2e^a+2\lim_{a\to-\infty}ae^a-2\lim_{a\to-\infty}e^a.
\end{aligned}$$

We have already computed before that

$$\lim_{a\to-\infty} e^a = 0 \quad \text{and} \quad \lim_{a\to-\infty} ae^a = 0.$$

We can also use L'Hopital twice to find that

$$\lim_{a\to-\infty} a^2e^a = \lim_{a\to-\infty} \frac{a^2}{e^{-a}} \overset{\frac{\infty}{\infty}}{=} \lim_{a\to-\infty} \frac{2a}{-e^{-a}} \overset{\frac{\infty}{\infty}}{=} \lim_{a\to-\infty} \frac{2}{e^{-a}} = 0.$$

Thus

$$\int_{-\infty}^0 x^2e^x dx = 2 \quad \text{and} \quad \int_{-\infty}^0 \frac{1}{2}x^2e^x dx = \frac{1}{2}\cdot 2 = 1.$$

Now let's compute the second piece.

$$\begin{aligned}
\int_0^\infty xe^{-x}dx &= \lim_{b\to\infty}\int_0^b xe^{-x}dx \\
&= \lim_{b\to\infty}(-x^2e^x - 2xe^{-x} - 2e^{-x})\Big|_0^b \\
&\quad \text{(used integration by parts twice)} \\
&= \lim_{b\to\infty}(-b^2e^{-b} - 2be^{-b} - 2e^{-b} + 2) \\
&= 2 - 2\lim_{b\to\infty} b^2e^{-b} - 2\lim_{b\to\infty} be^{-b} - 2\lim_{b\to\infty} e^{-b}.
\end{aligned}$$

We have found above that

$$\lim_{b\to\infty} e^{-b} = 0 \quad \text{and} \quad \lim_{b\to\infty} be^{-b} = 0,$$

and by applying L'Hopital twice we find that

$$\lim_{b\to\infty} b^2e^{-b} = \lim_{b\to\infty} \frac{b^2}{e^b} \overset{\frac{\infty}{\infty}}{=} \lim_{b\to\infty} \frac{2b}{e^b} \overset{\frac{\infty}{\infty}}{=} \lim_{b\to\infty} \frac{2}{e^b} = 0.$$

Thus,

$$\int_0^\infty x^2e^{-x}dx = 2 \quad \text{and} \quad \int_0^\infty \frac{1}{2}x^2e^{-x}dx = \frac{1}{2}\cdot 2 = 1,$$

thus

$$E(X^2) = 1 + 1 = 2.$$

Therefore,

$$\text{var}(X) = E(X^2) - E(X)^2 = 2 - 0^2 = 2.$$